Blind Sensor Calibration in Sparse Recovery Using Convex Optimization

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***Abstract*—We investigate a compressive sensing system in which the sensors introduce a distortion to the measurements in the form of unknown gains. We focus on *blind* calibration, using measures performed on a few unknown (but sparse) signals. We extend our earlier study on real positive gains to two generalized cases (signed real-valued gains; complex-valued gains), and show that the recovery of unknown gains together with the sparse signals is possible in a wide variety of scenarios. The simultaneous recovery of the gains and the sparse signals is formulated as a convex optimization problem which can be solved easily using off-the-shelf algorithms. Numerical simulations demonstrate that the proposed approach is effective provided that sufficiently many (unknown, but sparse) calibrating signals are provided, especially when the sign or phase of the unknown gains are not completely random.**

I. Introduction

Compressed sensing theory shows that K-sparse signals can be sampled at much lower rate than required by the Nyquist-Shannon theorem [1]. More precisely, if $\mathbf{x} \in \mathbb{C}^N$ is a K-sparse source vector then it can be captured by collecting only $M \ll N$ linear measurements

$$y_i = \mathbf{m}_i'\mathbf{x}, \qquad i = 1, \dots, M \tag{1}$$

In the above equation, $\mathbf{m}_1, \dots, \mathbf{m}_M \in \mathbb{C}^N$ are *known* measurement vectors, and $.'$ denotes the conjugate transpose operator. Under certain conditions on the measurement vectors, the signal can be accurately reconstructed by solving, e.g.,

$$\begin{aligned} \mathbf{x}^*_{\ell_1} = &\arg\min_{\mathbf{z}} \|\mathbf{z}\|_1 \\ &\text{subject to} \quad y_i = \mathbf{m}_i'\mathbf{z}, \quad i = 1, \dots, M \end{aligned}$$

where $\|\cdot\|_1$ denotes the ℓ_1-norm, which favors the selection of sparse signals among the ones satisfying the measurement constraints. It has been shown that the number of measurements needed for accurate recovery of $\mathbf{x}$ scales only linearly with K [1]. Note that the above minimization problem can easily be modified to handle the presence of additive noise on the measurements.

Unfortunately, in some practical situations, it is sometimes not possible to perfectly know the measurement vectors $\mathbf{m}_1, \dots, \mathbf{m}_M$. In many applications dealing with distributed sensors or radars, the location or intrinsic parameters of the sensors are not exactly known, which in turn results in unknown phase shifts and/or gains at each sensor [2], [3]. Similarly, applications with microphone arrays are shown to require calibration of each microphone to account for the unknown gain and phase shifts introduced [4]. Unlike additive perturbations in the measurement matrix, this multiplicative perturbation may introduce significant distortion if ignored [5], [6].

In this paper, we investigate the problem of estimating the unknown gains introduced by the sensors when multiple unknown but sparse input signals are measured. We extend the convex optimization approach dealing with positive real gains proposed in [7] to the case of signed real-valued and complex-valued gains which is more realistic from the application perspective. In addition to identifying the additional challenges introduced by the more difficult problem, we further demonstrate the performance of the proposed algorithms in cases where the unknown phase shifts (or sign changes) introduced by the sensors are not completely random.

II. Problem Definition

Suppose that the measurement system in (1) is perturbed by complex gains at each sensor i and there are multiple sparse input signals, $\mathbf{x}_l \in \mathbb{C}^N, \quad l = 1 \dots L$, applied to the system such that

$$y_{i,l} = d_i e^{j\theta_i}\mathbf{m}_i'\mathbf{x}_l \qquad i = 1 \dots M,\ \theta_i \in [0, 2\pi),\ d_i \in \mathbb{R}^+ \tag{2}$$

For a real valued system, the phase term $e^{j\theta_i}$ is replaced by ∓ 1 (or $\theta_i \in \{0, \pi\}$). We focus only on the noiseless case for the sake of simplicity.

It should be noted that, unlike the case with positive real gains, ignoring the unknown gains during recovery is not a viable option when dealing with signed real or complex gains even when the magnitude of the gains are constant. This is due to the significant distortion introduced by the change in sign (and phase). Therefore it is essential to employ a reconstruction approach that deals with the unknown gains.

In a traditional recovery strategy, one can enforce the sparsity of the input signals while enforcing the measurement constraints in (2). However, when dealing with unknown gains, the measurement constraints are non-linear with respect to the unknowns d_i and $\mathbf{x}_l$. This non-linearity can be dealt

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with by using an alternate minimization strategy where one iteratively estimates $\mathbf{x}$ while keeping d_i fixed and vice-versa [2]. However, the convergence of this alternating optimization to the global minimum is not guaranteed since there is a chance that the algorithm gets stuck in a local minimum.

A. Proposed Approach

The recovery of $\mathbf{x}_l$, $l = 1 \dots L$ and d_i, $i = 1 \dots M$ with convex optimization when $e^{j\theta_i}$ are known has been studied in [7]. In this paper, we extend the same approach to systems with signed real-valued and complex-valued gains. Therefore the term $d_i e^{j\theta_i}$ will henceforth simply be represented as $d_i \in \mathbb{R}$ for real-valued systems and $d_i \in \mathbb{C}$ for complex-valued systems.

As an alternative to the alternating non-linear optimization described above, the measurement equation (2) can be reorganized in a bi-linear fashion such that

$$y_{i,l}\tau_i = \mathbf{m}_i'\mathbf{x}_l \qquad i = 1 \dots M\ ,\ l = 1 \dots L \qquad (3)$$
$$\tau_i \triangleq \frac{1}{d_i}$$

assuming that $d_i \neq 0\ \forall i$. Consequently, one can attempt to recover the sparse signals and the gains with the convex optimization

$$\begin{array}{c}\mathbf{x}_1^*, \dots, \mathbf{x}_L^* \\ \tau_1^*, \dots, \tau_M^*\end{array} = \underset{\substack{\mathbf{z}_1,\dots,\mathbf{z}_L \\ t_1,\dots,t_M}}{\arg\min} \sum_{n=1}^{L} \|\mathbf{z}_n\|_1 \qquad (4)$$
$$\text{subject to} \quad y_{i,l}t_i = \mathbf{m}_i'\mathbf{z}_l \qquad \begin{array}{l} l = 1, \dots, L \\ i = 1, \dots, M \end{array}$$
$$\sum_{n=1}^{M} t_n = c$$

for an arbitrary constant $c > 0$. The actual gains can be set as $d_i^* = \frac{1}{\tau_i^*}$ after the optimization. Note that even though the minimized objective function is equivalent to the alternating non-linear optimization, the problems with local minimums are now eliminated thanks to the convexity of the formulation.

We can make several observations regarding the optimization in (4):

1) The constraint $\sum_n t_n = c$ ensures that the trivial solution ($\tau_i = 0,\ \mathbf{x}_l = 0,\ \forall i, l$) is excluded from the solution set.
2) The constraint $\sum_n t_n = c$ also excludes the solutions where the sum of the gains are zero. When dealing with signed real or complex valued gains, this may result in excluding the actual solution in rare cases where the sought out gains actually sum up to zero. However, the probability of encountering this phenomena in real applications is often infinitesimally small. For the applications in which this possibility is higher, an alternative approach to deal with this case is discussed in Section III.
3) The measurement constraints are satisfied up to a global scale factor (and phase shift for complex signals), therefore the constant c can be set arbitrarily. Unfortunately, the global scale (and phase) factor cannot be determined with the given optimization approach, although this is often not an issue in practical systems.
4) The successful recovery of the gains and the signals require availability of more than one input signal ($L > 1$). Although this may seem like a restriction, acquiring data from multiple sources is often straightforward in many application fields.

III. Experimental Results

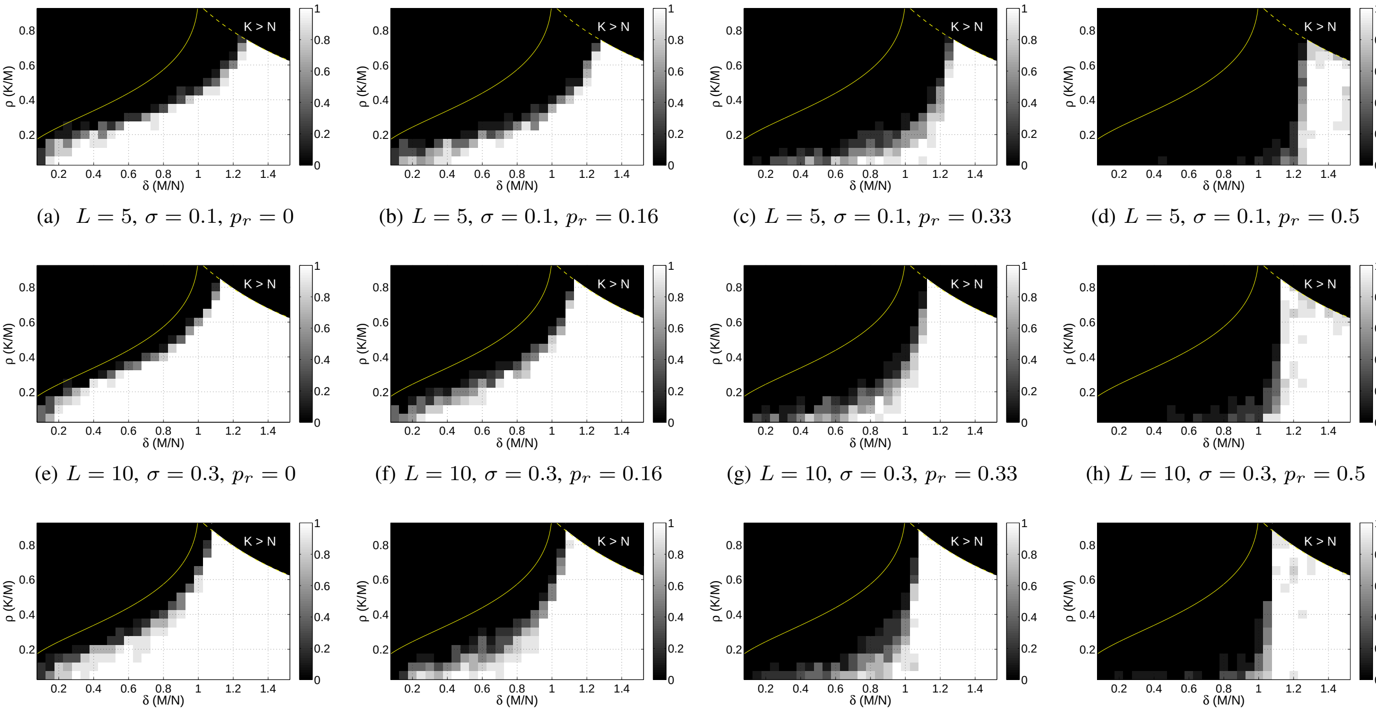
In order to test the performance of the proposed algorithm, phase transition curves as in the compressed sensing recovery are plotted for a signal size $N = 100$ with the measurement vectors, $\mathbf{m}_i$, and all the non zero entries in the input signals, $\mathbf{x}_l$, randomly generated from an i.i.d. normal distribution. The positions of the non-zero coefficients of the input signals, $\mathbf{x}_l$, are chosen uniformly at random in $\{1, \dots, N\}$. The magnitude of the gains were generated using $|d_i| \sim \exp(\mathcal{N}(0, \sigma^2))$, where σ is the parameter governing the amplitude of decalibration. For real valued experiments, the sign of the gains are randomly assigned such that the probability, p_r, of setting a negative gain is adjusted to be $p_r \in \{0, 0.16, 0.33, 0.5\}$. Similarly for complex valued gains, the phase of the gains are chosen uniformly at random from the range $[0,\ 2\pi p_c)$ where $p_c \in \{0, 0.33, 0.66, 1\}$. Note that the parameters p_r and p_c determine the scale of ambiguity in the signs and phases where maximum possible ambiguity is observed when $p_r = 0.5$ and $p_c = 1$ respectively.

The signals (and the gains) are recovered for different amount of decalibration amplitude ($\sigma = 0.1,\ 0.3,\ 1$) with sufficiently high number of input signals ($L = 5,\ 10,\ 20$ respectively). The proposed optimization in (4) is performed using an ADMM [9] based algorithm. The perfect reconstruction criteria is selected as $\frac{1}{L}\sum_l \mu(\mathbf{x}_l, \mathbf{x}_l^*) > 0.9999$, where the absolute correlation factor $\mu(\cdot,\cdot)$ is defined as

$$\mu(\mathbf{x}_1, \mathbf{x}_2) \triangleq \frac{|\mathbf{x}_1'\mathbf{x}_2|}{\|\mathbf{x}_1\|_2\|\mathbf{x}_2\|_2} \qquad (5)$$

so that the global phase and scale difference between the source and recovered signals is ignored.

The probability of recovery (computed through 10 independent simulations for each set of parameters) of the proposed method with respect to $\delta \triangleq M/N$ and $\rho \triangleq K/M$ are shown in Figures 1 and 2 for real valued and complex valued systems respectively. The first thing to notice from the results is that the performance for $p_r = 0$ (or $p_c = 0$) is consistent with the results presented in [7] as expected. The effect of increasing sign (or phase) ambiguity can be observed in the results as p_r (or p_c) increases. Although the performance is acceptable for p_r as high as 0.33 (p_c up to 0.66), there is a significant degradation when dealing very high sign (or phase) ambiguity such that signal recovery is impossible regardless of the sparsity, unless the measurement system is overcomplete ($M > N$). This phenomena can best be observed in the last row of complex-valued results, Figures 2(m)-2(p), where the number of input signals is very large ($L = 50$) with respect to



(a) $L = 5$, $\sigma = 0.1$, $p_r = 0$ (b) $L = 5$, $\sigma = 0.1$, $p_r = 0.16$ (c) $L = 5$, $\sigma = 0.1$, $p_r = 0.33$ (d) $L = 5$, $\sigma = 0.1$, $p_r = 0.5$

(e) $L = 10$, $\sigma = 0.3$, $p_r = 0$ (f) $L = 10$, $\sigma = 0.3$, $p_r = 0.16$ (g) $L = 10$, $\sigma = 0.3$, $p_r = 0.33$ (h) $L = 10$, $\sigma = 0.3$, $p_r = 0.5$

(i) $L = 20$, $\sigma = 1$, $p_r = 0$ (j) $L = 20$, $\sigma = 1$, $p_r = 0.16$ (k) $L = 20$, $\sigma = 1$, $p_r = 0.33$ (l) $L = 20$, $\sigma = 1$, $p_r = 0.5$

Fig. 1: The probability of perfect recovery in the real valued system for $N = 100$ with respect to $\delta \triangleq M/N$ and $\rho \triangleq K/M$. The solid yellow line indicates the Donoho-Tanner phase transition curve for fully calibrated compressed sensing recovery [8]. The dashed yellow line indicates the boundary to the region where $K > N$. Each row of figures display the change in recovery performance with increasing sign ambiguity from left to right for a fixed set of L and σ.

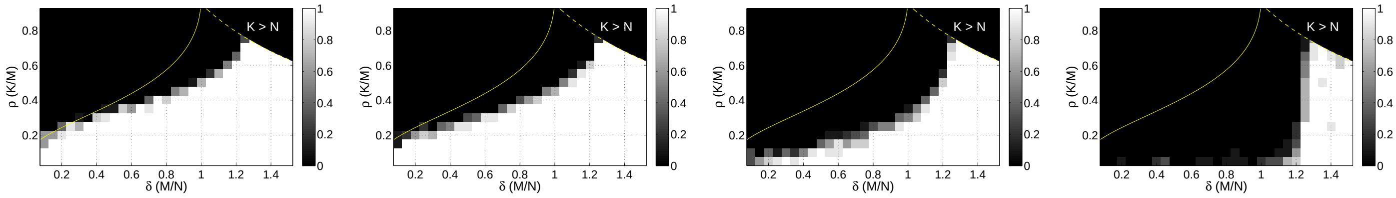
the variance in the gain magnitudes ($\sigma = 0.1$). The degradation in the results can be attributed to the significant increase in the contamination of the information in the measurements as the sign or phase ambiguity increases. Therefore recovery becomes possible only when there are sufficient number of measurements to overcome the high distortion. For the maximally ambiguous case ($p_r = 0.5$, $p_c = 1$), this is only possible for $M > N$. Even though this is a drawback of the presented approach, it should be noted that in many practical systems the sign (or phase) ambiguity is often not as severe as fully random, but within a limited range. Therefore the presented algorithm can still be applied in various scenarios.

As an alternative to the proposed method in this paper, a phase calibration algorithm (in which gain magnitudes are assumed to be known) that can recover the sparse signals along with the unknown phases distributed within the entire $[0, 2\pi)$ range has been presented in [10], [11]. This approach for phase calibration can be combined with the proposed method in this paper in order to recover signed real-valued or complex-valued gains with maximum sign and phase ambiguity. It is also possible to use this combined approach for signal recovery in applications where the sum of the gains are likely to be zero.

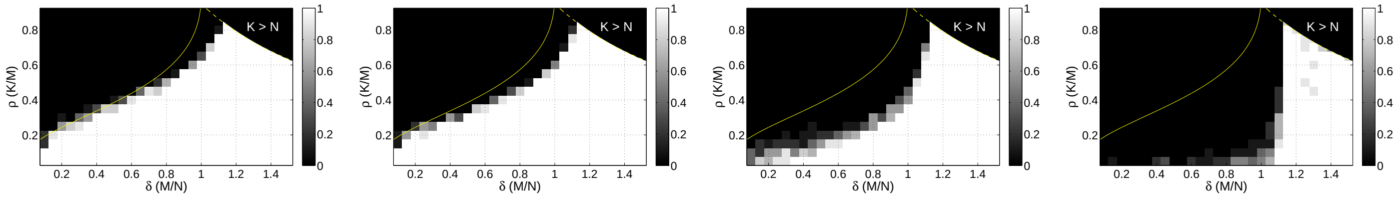
IV. Conclusions and Future Work

In this paper, we have investigated the problem of estimating the unknown gains at each measurement sensor along with sparse input signals in a compressed sensing measurement system. We have extended the use of convex recovery strategy suggested for positive real gains to the more general cases of signed real-valued and complex-valued gains, and demonstrated the change of recovery performance with increasing sign and phase ambiguity.

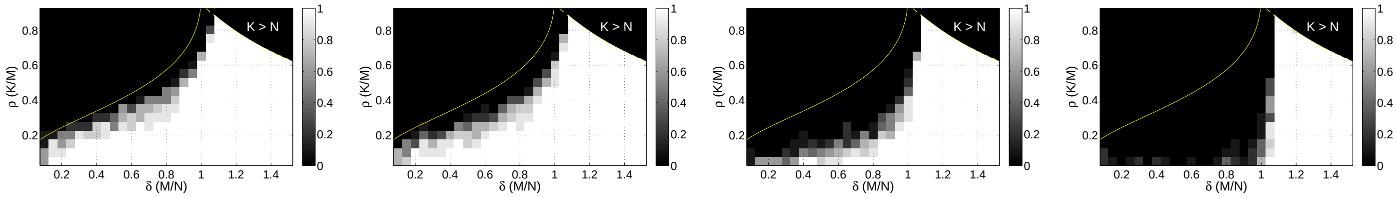
The performance of the proposed algorithm is shown to be approaching to that of the unperturbed compressed sensing recovery when there are sufficient number sparse input signals unless the distribution of the sign changes or the phase shifts are maximally varying among the sensors. This drawback of the proposed algorithm can still be ignored for many application fields in which the ambiguity in the sign changes or the phase shifts at the sensors are within a limited range. For other applications, it is possible to combine the proposed method with other approaches employed for phase calibration to improve the recovery performance which is considered as a future work. The theoretical justification of the limitation of the proposed method for maximum sign and phase ambiguity is also a work in progress.



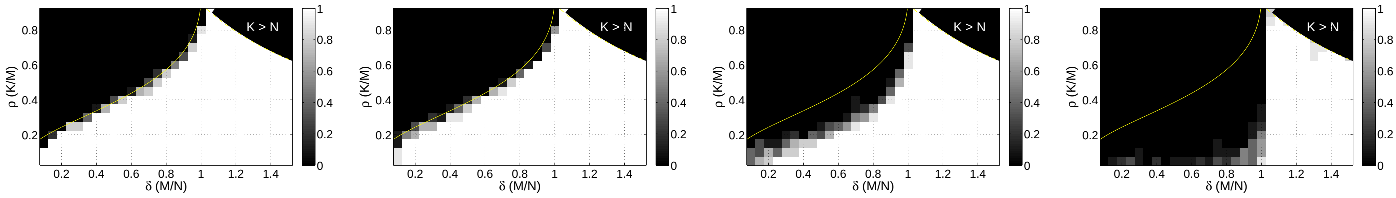
(a) $L = 5$, $\sigma = 0.1$, $p_c = 0$ (b) $L = 5$, $\sigma = 0.1$, $p_c = 0.33$ (c) $L = 5$, $\sigma = 0.1$, $p_c = 0.66$ (d) $L = 5$, $\sigma = 0.1$, $p_c = 1$



(e) $L = 10$, $\sigma = 0.3$, $p_c = 0$ (f) $L = 10$, $\sigma = 0.3$, $p_c = 0.33$ (g) $L = 10$, $\sigma = 0.3$, $p_c = 0.66$ (h) $L = 10$, $\sigma = 0.3$, $p_c = 1$



(i) $L = 20$, $\sigma = 1$, $p_c = 0$ (j) $L = 20$, $\sigma = 1$, $p_c = 0.33$ (k) $L = 20$, $\sigma = 1$, $p_c = 0.66$ (l) $L = 20$, $\sigma = 1$, $p_c = 1$



(m) $L = 50$, $\sigma = 0.1$, $p_c = 0$ (n) $L = 50$, $\sigma = 0.1$, $p_c = 0.33$ (o) $L = 50$, $\sigma = 0.1$, $p_c = 0.66$ (p) $L = 50$, $\sigma = 0.1$, $p_c = 1$

Fig. 2: The probability of perfect recovery in the complex valued system for $N = 100$ with respect to $\delta \triangleq M/N$ and $\rho \triangleq K/M$. The solid yellow line indicates the Donoho-Tanner phase transition curve for fully calibrated compressed sensing recovery [8]. The dashed yellow line indicates the boundary to the region where $K > N$. Each row of figures display the change in recovery performance with increasing phase ambiguity from left to right for a fixed set of L and σ. The last row, (m)-(p) shows the performance limit for very high L.

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